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Note

No-hole $L(2,1)$-colorings

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Abstract

An $L(2,1)$-*coloring* of a graph G is a coloring of G's vertices with integers in $\{0,1,\dots,k\}$ so that adjacent vertices' colors differ by at least two and colors of distance-two vertices differ. We refer to an $L(2,1)$-coloring as a coloring. The *span* $\lambda(G)$ of G is the smallest k for which G has a coloring, a *span coloring* is a coloring whose greatest color is $\lambda(G)$, and the *hole index* $\rho(G)$ of G is the minimum number of colors in $\{0,1,\dots,\lambda(G)\}$ not used in a span coloring. We say that G is *full-colorable* if $\rho(G)=0$. More generally, a coloring of G is a *no-hole coloring* if it uses all colors between 0 and its maximum color. Both colorings and no-hole colorings were motivated by channel assignment problems. We define the *no-hole span* $\mu(G)$ of G as ∞ if G has no no-hole coloring; otherwise $\mu(G)$ is the minimum k for which G has a no-hole coloring using colors in $\{0,1,\dots,k\}$.

Let n denote the number of vertices of G, and let Δ be the maximum degree of vertices of G. Prior work shows that all non-star trees with $\Delta \geqslant 3$ are full-colorable, all graphs G with $n=\lambda(G)+1$ are full-colorable, $\mu(G) \leqslant \lambda(G)+\rho(G)$ if G is not full-colorable and $n \geqslant \lambda(G)+2$, and G has a no-hole coloring if and only if $n \geqslant \lambda(G)+1$. We prove two extremal results for colorings. First, for every $m \geqslant 1$ there is a G with $\rho(G)=m$ and $\mu(G)=\lambda(G)+m$. Second, for every $m \geqslant 2$ there is a connected G with $\lambda(G)=2m$, $n=\lambda(G)+2$ and $\rho(G)=m$.

Keywords: Distance-two colorings; No-hole colorings; Channel assignment problems

1. Introduction

Let $\mathcal{G}$ be the family of simple graphs with non-empty, finite vertex sets. A *coloring* of $G=(V,E)$ in $\mathcal{G}$ is a map $f: V \to \mathbb{Z}$. It is a *proper vertex coloring* if $f(x) \neq f(y)$

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whenever $\{x, y\} \in E$, and a *no-hole coloring* if $f(V)$ is a set of consecutive integers. A *hole* in f is an integer i such that $\min f(V) < i < \max f(V)$ and $i \notin f(V)$.

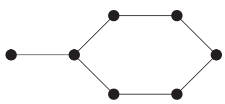
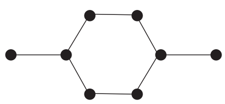
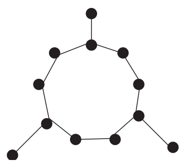
Although every $G \in \mathscr{G}$ has a proper vertex coloring with no holes, this need not be true for more restrictive colorings. A case in point for $r \geqslant 1$ is an $(r+1)$-distant coloring which requires $|f(x) - f(y)| \geqslant r+1$ whenever $\{x, y\} \in E$. Motivated by channel assignment problems, Roberts [10] and Sakai and Wang [12] study the question of what graphs have $(r+1)$-distant colorings that are no-hole colorings. The present paper does the same for $L(2,1)$-*colorings*, which are colorings in which adjacent vertices' colors differ by at least two and colors of distance-two vertices differ. *We refer henceforth to an* $L(2,1)$-*coloring as a coloring*.

Colorings were first studied extensively by Yeh [14] and Griggs and Yeh [6] as a generalization of T-colorings formulated by Hale [7] for the channel assignment problem. Other studies of colorings include [1–5,8,9,11,13]. Much of this work focuses on the *span* $\lambda(G)$ of $G \in \mathscr{G}$, defined as the minimum k for which G has a coloring with $f(V) \subseteq \{0, 1, \ldots, k\}$. We refer to a coloring with $\min f(V) = 0$ and $\max f(V) = \lambda(G)$ as a *span coloring*, and define the *hole index* $\rho(G)$ of G as the minimum, over all span colorings of G, of the number of holes in the coloring.

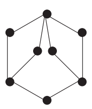
We say that G is *full-colorable* if $\rho(G) = 0$. The following proposition recalls results on full-colorability in [2] along with an easy implication (iii) of Theorem 1.1 in Georges et al. [5]. Let $n = |V|$ and let Δ denote a graph's maximum vertex degree.

Proposition 1. *The following are full-colorable*:

(i) *all connected* $\Delta = 2$ *graphs except path* P_3 *and cycles* C_3, C_4 *and* C_6;
(ii) *all trees with* $\Delta \geqslant 3$ *except stars*;
(iii) *all graphs with* $n = \lambda + 1$;
(iv) *all connected non-tree graphs with* $(\Delta, \lambda) = (3, 4)$ *and* $n \geqslant 6$ *except*

 ,  *and*  ;

(v) *all connected non-tree graphs with* $(\Delta, \lambda) = (4, 5)$ *and* $n \in \{7, 8\}$ *except*

 .

The present note considers another parameter for colorings. The *no-hole span* $\mu(G)$ of $G \in \mathscr{G}$ is ∞ if G has no no-hole coloring; otherwise $\mu(G)$ is the minimum k for which G has a no-hole coloring using colors in $\{0, 1, \ldots, k\}$. If G is not full-colorable then our definitions give $\rho(G) \geqslant 1$ and $\mu(G) \geqslant \lambda(G)+1$. It is easily verified that the four graphs pictured in Proposition 1 have $\mu(G)=\lambda(G)+1$.

Because $|\{0,1,\ldots,\lambda(G)\}| = \lambda(G)+1$, G is not full-colorable and $\mu(G)=\infty$ if $n \leqslant \lambda(G)$. This is true, for example, for every star with $n \geqslant 2$. We also have the following straightforward consequence of Lemmas 2.2 and 2.3 in [5].

Proposition 2. *If $G \in \mathscr{G}$ is not full-colorable and $n \geqslant \lambda(G)+2$, then $\mu(G) \leqslant \lambda(G) + \rho(G)$.*

A corollary of these observations and Proposition 1(iii) is that G has a no-hole coloring if and only if $n \geqslant \lambda(G)+1$.

Our purpose here is to add two extremal results to the preceding. The first says that a large hole index ρ can accompany a large no-hole span μ. In particular, equality is possible in the conclusion of Proposition 2. We prove this in Section 2.

Theorem 1. *For every $m \geqslant 1$ there is a $G \in \mathscr{G}$ with $\rho(G)=m$ and $\mu(G)=\lambda(G)+m$.*

Our second result says that for each $m \geqslant 2$ there is a connected graph with span $2m$, $n=2m+2$, and hole index m, which is the maximum possible in view of the observation in [5] that if H denotes the set of holes in a span coloring with as few holes as possible then $0, \lambda \notin H$ and if $h \in H$ then neither $h-1$ nor $h+1$ is in H. The proof in Section 3 uses graphs with $\mu(G)=\lambda(G)+1$.

Theorem 2. *For each $m \geqslant 2$ there is a connected graph G with $\lambda(G)=2m$, $n=\lambda(G)+2$ and $\rho(G)=m$.*

2. Graphs with large no-hole spans

To prove Theorem 1 we construct a graph G_m for each $m \geqslant 2$ that has $\rho(G_m)=m$ and $\mu(G_m)=\lambda(G_m)+m$.

For each $m \geqslant 2$ let G_m be the graph with $n=5(m+1)$ that consists of disjoint copies A and B of K_{2m+2}, a K_{m+1} disjoint from $A \cup B$ that we denote by C, and assorted other edges between C and $A \cup B$ as follows. Denote the vertices by

$a_0, a_2, a_4, \ldots, a_{4m+2}$ for A,

$b_0, b_2, b_4, \ldots, b_{4m+2}$ for B,

$c_1, c_5, c_9, \ldots, c_{4m+1}$ for C.

The other edges are between c_1 and each of $a_4, a_6, \ldots, a_{4m+2}$; between c_5 and each of a_0 and a_2; between c_5 and each of $b_8, b_{10}, \ldots, b_{4m+2}$; and between c_j and b_k if $j \geqslant 9$ and $|j-k| \geqslant 2$. Fig. 1 pictures G_2 and G_3.

Note that each $c_k \in C$ has exactly one edge to $\{a_j, b_j\}$ when $|j-k| \geqslant 2$, and has no edges to $\{a_{k-1}, a_{k+1}, b_{k-1}, b_{k+1}\}$. Each vertex in A has one edge to C and each vertex

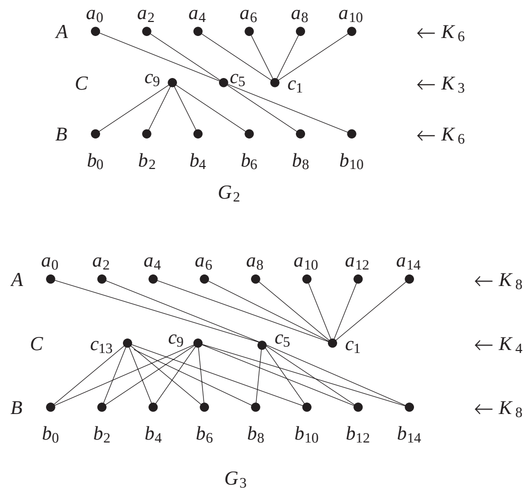


Fig. 1. Graphs for $\mu(G_m) = \lambda(G_m) + m$. The edges between every two vertices within each row are *not* shown.

in B has exactly $m - 1$ edges to C. It follows that if $x \in A$ and $y \in B$, and if x and y do not have edges to the same vertex in C, then together they have edges to all but one of the $m + 1$ vertices in C.

By construction, the coloring of G_m that assigns color i to a_i, color j to b_j, and color k to c_k defines a coloring. Because $\lambda(K_{2m+2}) = 4m + 2$, $\lambda(G_m) = 4m + 2$ and every span coloring assigns precisely the colors in $\{0, 2, 4, \ldots, 4m + 2\}$ to each of A and B. There are $4m + 3$ integers between 0 and $4m + 2$ inclusive. Hence any feasible assignment of colors between 0 and $4m + 2$ to C has at least

$$4m + 3 - (2m + 2) - (m + 1) = m$$

holes. The coloring given shows that $\rho(G_m) = m$.

Suppose f is a no-hole coloring of G_m with $f(A \cup B \cup C) = \{0, 1, \ldots, \mu\}$. If color k is used more than once by f, then, since every vertex in C has distance at most 2 from all vertices of G_m, k is not used for C. Hence, k can be used only for one vertex in A and one in B, and the two cannot have edges to the same vertex in C. Suppose $1 \leqslant k \leqslant \mu - 1$ and $f(x) = f(y) = k$ for $x \in A$ and $y \in B$. Then x and y have edges to exactly m vertices in C, so none of those m has a color in $\{k - 1, k + 1\}$. Because no vertex in $A \cup B$ has a color in $\{k - 1, k + 1\}$, we are left with only one vertex whose color can be in $\{k - 1, k + 1\}$. This contradicts our no-hole supposition.

We conclude that only 0 and μ can be used for more than one vertex by f, and each can be assigned to no more than two vertices. If 0 and μ are in fact assigned by f to a total of four vertices, then $5m + 1$ vertices remain uncolored and each must get a different color in $\{1, 2, \ldots, \mu - 1\}$. It is easily verified that such a no-hole coloring exists, so $\mu(G_m) = \mu = 5m + 2 = \lambda(G_m) + m$. A specific no-hole coloring of this type follows:

$$C : f(c_1) = 5m+1, f(c_5) = 5(m-1)+1, f(c_9) = 5(m-2)+1, \ldots,$$
$$f(c_{4m+1}) = 1;$$
$$A : f(a_0) = 5m+2, f(a_2) = 2, \text{ and}$$
$$(f(a_4), f(a_6), \ldots, f(a_{4m+2})) = (0, 4, 7, 9, 12, 14, 17, 19, \ldots).$$
$$B : (f(b_{4m+2}), f(b_{4m}), \ldots, f(b_0)) = (0, 3, 5, 8, 10, 13, 15, 18, \ldots).$$

The f sequences shown for B and A alternate the colors from 3 through $5m$, skipping colors used for C.

3. Maximum hole indices

Two lemmas guide our construction of connected graphs that maximize ρ for a given λ. Let f denote a span coloring of $G=(V,E)$ with as few holes as possible. Also let

$$V_j = \{v \in V : f(v) = j\}, \quad v_j = |V_j|, \quad H = \{0 \leqslant j \leqslant \lambda : v_j = 0\}$$

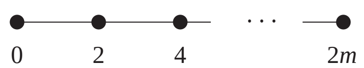
so that $\rho(G) = |H|$.

Lemma 1. 0, $\lambda \notin H$; *if* $h \in H$ *then* $h-1 \notin H$ *and* $h+1 \notin H$.

Lemma 2. *Suppose* $n \geqslant \lambda+1$ *and* $h \in H$. *Then* $v_{h-1} = v_{h+1} \geqslant 2$ *and there is a bijection* $g : V_{h-1} \to V_{h+1}$ *such that the set of edges of* E *within* $V_{h-1} \cup V_{h+1}$ *is* $\{\{v, g(v)\} : v \in V_{h-1}\}$.

We noted Lemma 1 prior to Theorem 2. Our proof of Lemma 2, which is similar to proofs in [5] apart from the matching edges feature, is straightforward and will be omitted.

Now suppose that $\lambda(G) = 2m \geqslant 4$. Then $\rho(G) \leqslant m$ by Lemma 1, and if $\rho(G) = m$ then every span coloring of G must have hole set $\{1, 3, 5, \ldots, 2m-1\}$. It follows from Lemma 2 for $\rho(G) = m$ that every span coloring of G partitions G into $t \geqslant 2$ colored subgraphs

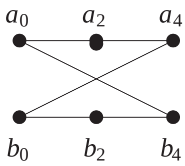


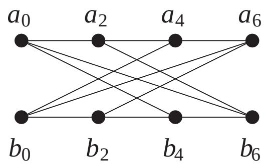
with no other edges between vertices whose colors differ by 2. Our ensuing construction takes $t = 2$.

Given $m \geqslant 2$, let F_m be the graph with $2m+2$ vertices $a_0, a_2, a_4, \ldots, a_{2m}, b_0, b_2, b_4, \ldots, b_{2m}$ and edge set

$$\{\{a_i, a_{i+2}\} : i = 0, 2, \ldots, 2m-2\} \cup \{\{b_i, b_{i+2}\} : i = 0, 2, \ldots, 2m-2\}$$
$$\cup \{\{a_i, b_j\} : |i-j| \geqslant 4\}.$$

Thus

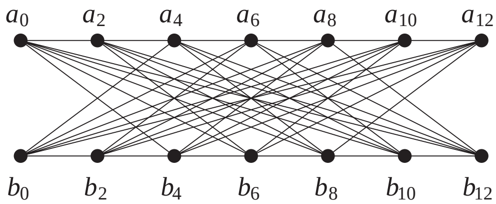
$F_2 =$ 

and $F_3 =$ 

Because $F_2 = C_6$ and every span coloring of C_6 uses only colors 0, 2 and 4, $\rho(F_2) = 2$. Assume $m \geqslant 3$ henceforth. The bulk of the proof that $\lambda(F_m) = 2m$ and $\rho(F_m) = m$ is borne by the following lemma.

Lemma 3. *For every $m \geqslant 3$ and every coloring of F_m, at most two vertices have any given color, and if color k is used for two vertices then color k' is a hole when $|k' - k| = 1$.*

Proof. We defer the proof for $m \leqslant 5$ momentarily and begin with $m \geqslant 6$. Graph F_6 is

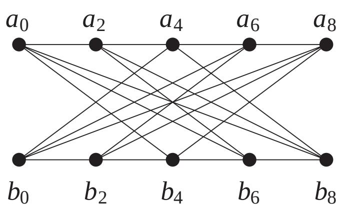


Inspection shows that the distance between vertices x and y is 1 or 2 except when $\{x, y\} = \{a_i, b_i\}$ for $i = 0, 2, \ldots, 12$. Hence if color k is used more than once then it is used exactly twice with $f(a_i) = f(b_i) = k$ for some i. Inspection also shows that every vertex not in $\{a_i, b_i\}$ is adjacent to either a_i or b_i. It follows that if $f(a_i) = f(b_i) = k$ then no vertex has color k' when $|k' - k| = 1$.

The same conclusions clearly hold for $m \geqslant 7$, so Lemma 3 holds for all $m \geqslant 6$.

Suppose $m = 3$. The preceding picture for F_3 reveals that the distance between x and y exceeds 2 if and only if $\{x, y\}$ is an $\{a_i, b_i\}$ or $\{a_0, a_6\}$ or $\{b_0, b_6\}$. Moreover, for each of these pairs, every other vertex is adjacent to one member of the pair. We conclude that at most two vertices can have the same color and, when they do, the adjacent colors cannot be used for any vertex.

The picture for F_4,



shows that the distance between x and y exceeds 2 if and only if $\{x, y\}$ is an $\{a_i, b_i\}$ or one of $\{a_0, a_6\}$, $\{b_0, b_6\}$, $\{a_2, a_8\}$ and $\{b_2, b_8\}$. In each case, every other vertex is adjacent to one member of the pair, so Lemma 3 holds for $m = 4$.

When $m = 5$, the only pairs besides the $\{a_i, b_i\}$ that can have one color are $\{a_2, a_8\}$ and $\{b_2, b_8\}$, and in each case every other vertex is adjacent to one member of the pair. The conclusion of the lemma therefore holds when $m=5$ and the proof is complete. □

For any coloring of F_m, let c_k be the number of vertices with color k and let $c = (c_0, c_1, c_2, \ldots, c_p)$ where $c_p > 0$ and $c_k = 0$ for all $k > p$. Note that $\Sigma c_k = 2m + 2$. By Lemma 3, $c_k \leqslant 2$, and if $c_k = 2$ then $c_j = 0$ when $|j - k| = 1$. The coloring f of F_m with $f(a_i) = f(b_i) = i$ for $i = 0, 2, \ldots, 2m$ has $c = (2, 0, 2, 0, \ldots, 0, 2)$ and shows that $\lambda(F_m) \leqslant 2m$. If c has 1's, then each maximal run of r consecutive 1's can be replaced by $2020 \cdots 201$ with r terms when r is odd, and by $2020 \cdots 2$ with $r - 1$ terms when r is even. This results in a new $c' = (c'_0, c'_1, c'_2, \ldots, c'_q)$ with $q \leqslant p$, no two consecutive terms positive, and $\Sigma c'_k = 2m + 2$. Clearly $q \geqslant 2m$, which shows that $\lambda(F_m) = 2m$. Moreover, $q = 2m$ if and only if $c'' = (2, 0, 2, 0, \ldots, 0, 2)$, which shows that every span coloring has $c_1 = c_3 = \cdots = c_{2m-1} = 0$, and hence that $\rho(F_m) = m$.

Acknowledgements

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